The 334-Triangle Graph of $SL_3(\mathbb{Z})$

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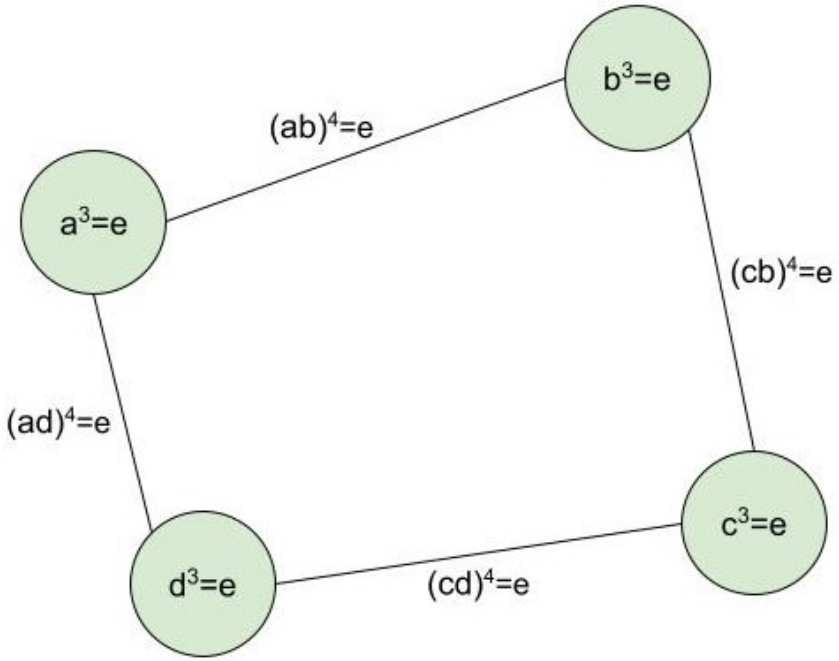
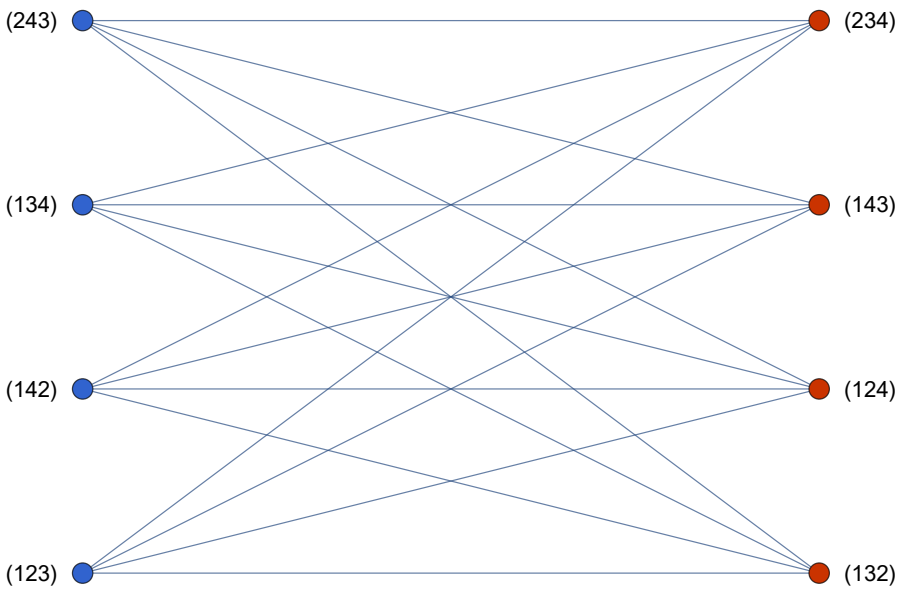
What is a 334-Triangle Graph?

Definition

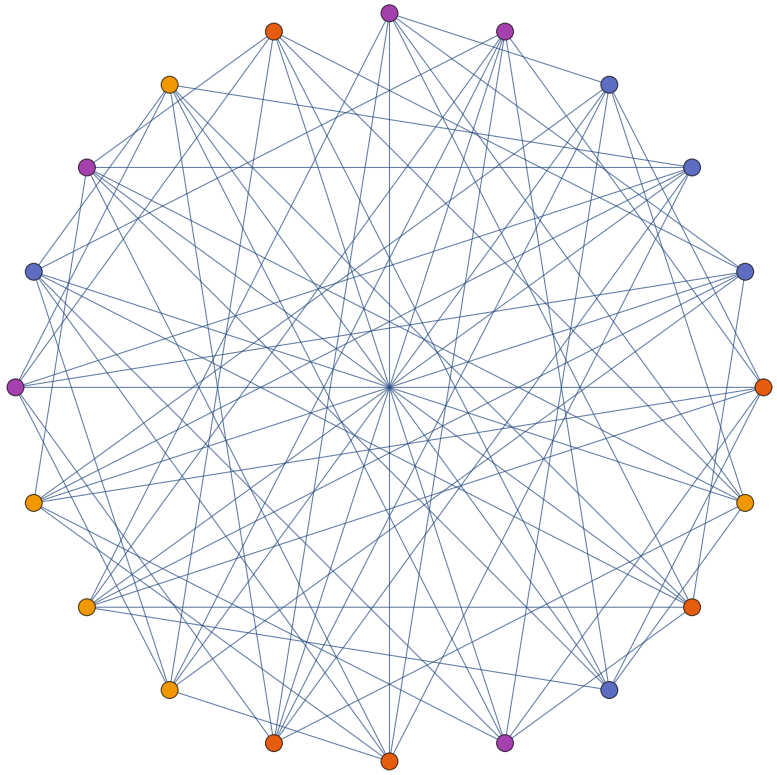
For any group G, let $\Delta 334(G)$ be the graph whose vertices are the elements $a \in G$ such that $a^3 = e$, in which there is an edge between two vertices a and b if and only if $(ab)^4 = e$.



What do 334-Triangle Graphs look like?



The non-identity component of $\Delta 334(S_4)$.

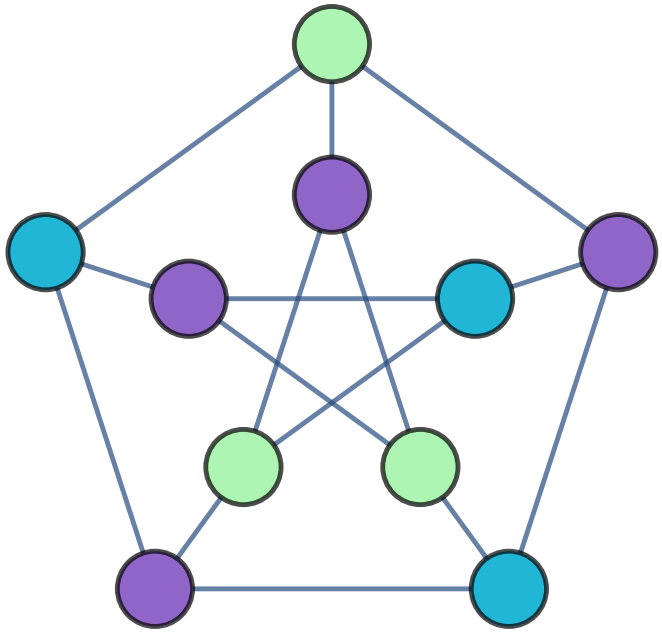


The non-identity component of $\Delta 334(S_5)$.

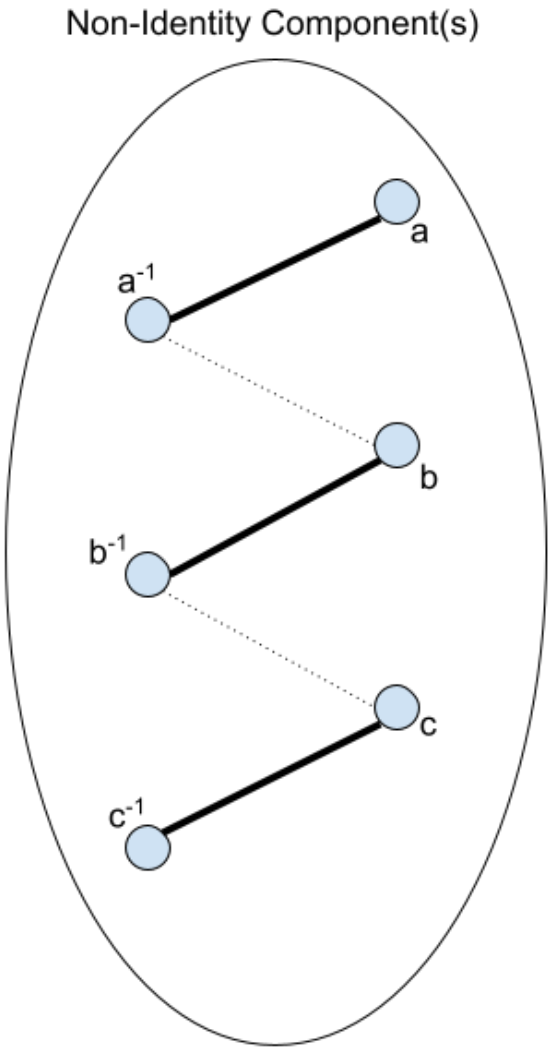
What is the Chromatic Number of a Graph?

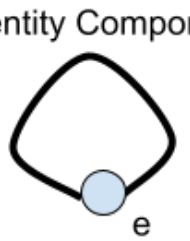
Definition

The chromatic number of a graph is the smallest number of colors required to color the vertices of the graph such that no two adjacent vertices are the same color.



What about Abelian Groups?





For Abelian groups, only thick edges exist

Why do we Care About 334-Triangle Graphs?

Definition

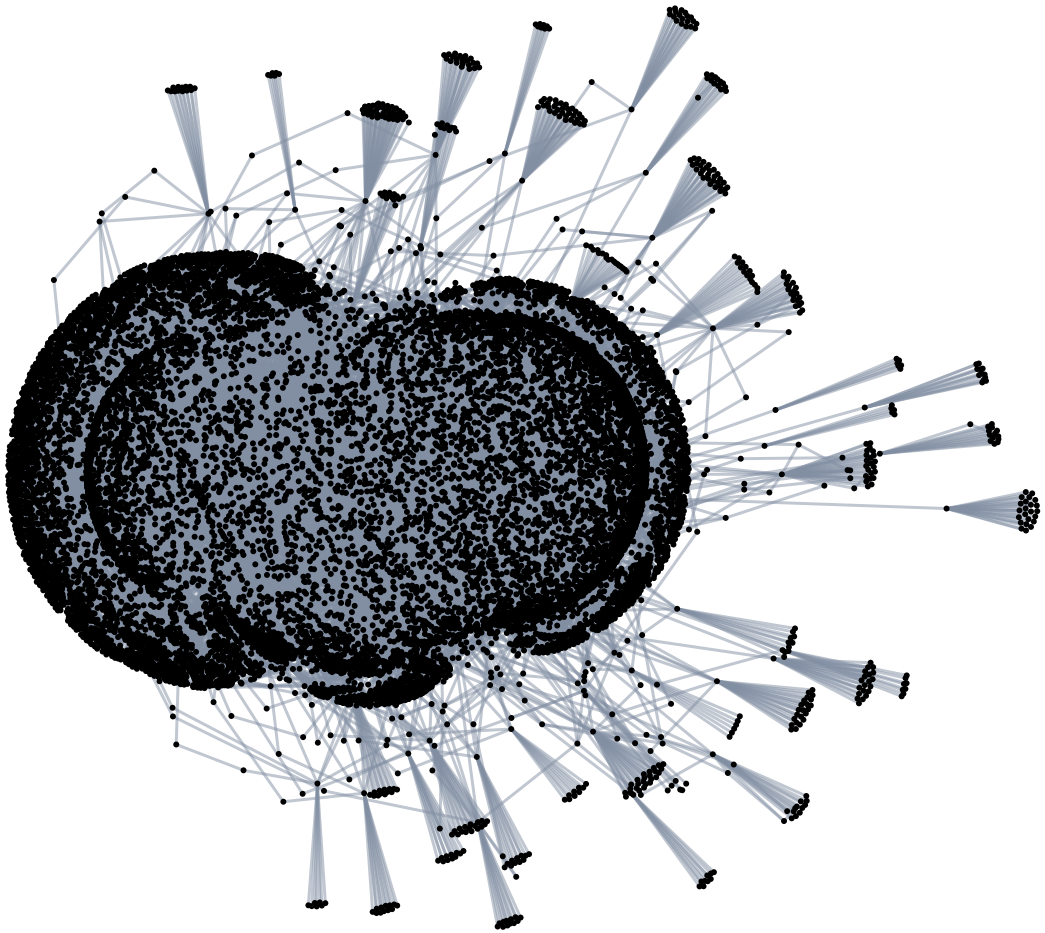
A thin group is a subgroup of $SL_n(\mathbb{Z})$ which has infinite index and is Zariski dense.

Some representations of $T = \langle a, b \mid a^3 = b^3 = (ab)^4 = e \rangle$ in $SL_3(\mathbb{Z})$ are thin, while others are not.

What do we Know About $\Delta 334(SL_3(\mathbb{Z}))$?

So far...

- $\sim 25,000$ vertices
- Chromatic number $= 4$
- No cliques greater than 3
- Cycles of varying lengths



How can we Bound the Chromatic Number?

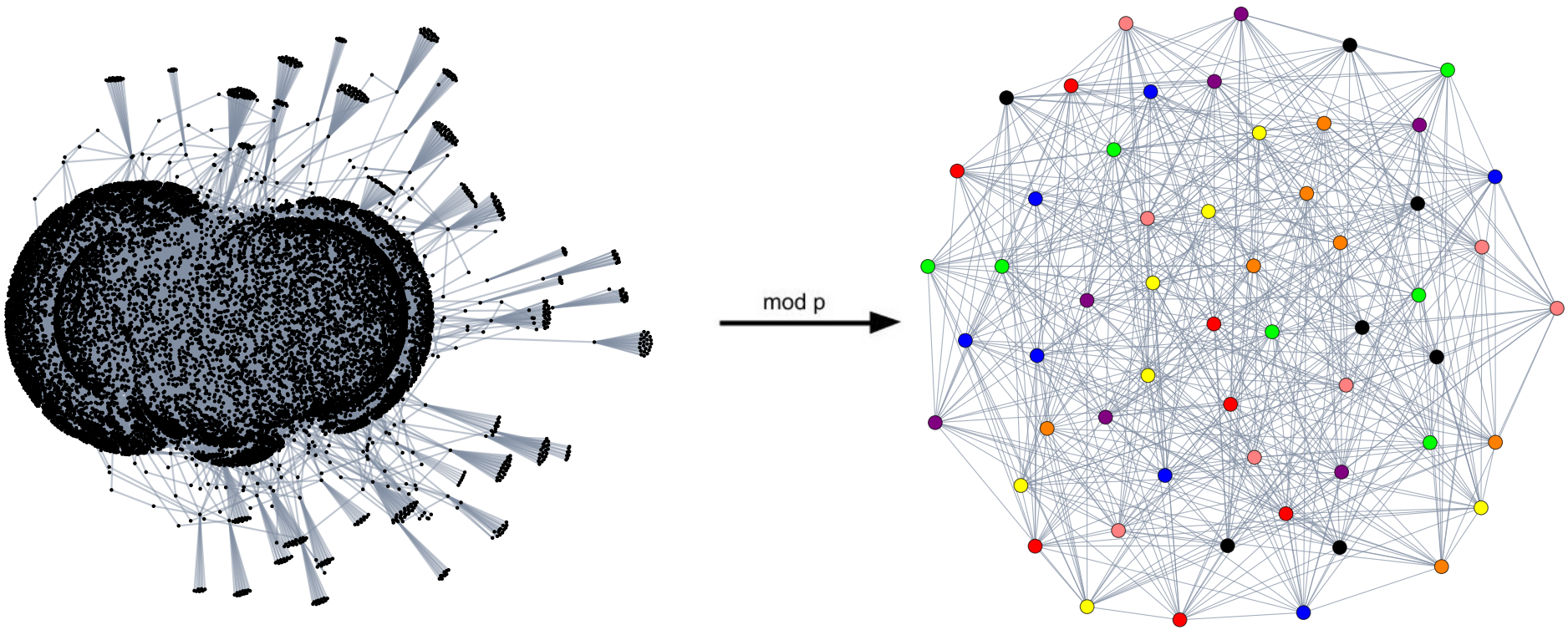
Theorem

The chromatic number of $\Delta 334(SL_3(\mathbb{Z}))$ *is at most the chromatic number of* $\Delta 334(SL_3(\mathbb{Z}/p\mathbb{Z}))$ *where* p *is prime.*

Proof outline:

- No order three matrix in $SL_3(\mathbb{Z})$ reduces to the identity mod p
- If two vertices are adjacent in $\Delta 334(SL_3(\mathbb{Z}))$, their images are adjacent in $\Delta 334(SL_3(\mathbb{Z}/p\mathbb{Z}))$
- Lift proper coloring from $\Delta 334(SL_3(\mathbb{Z}/p\mathbb{Z}))$ to $\Delta 334(SL_3(\mathbb{Z}))$

Upper Bound on the Chromatic Number of $\Delta 334(SL_3(\mathbb{Z}))$



Using $p = 2$: $4 \leq \chi(\Delta 334(SL_3(\mathbb{Z}))) \leq 8$

Questions we still have about 334-Triangle Graphs

Conjectures about $\Delta 334(SL_3(\mathbb{Z}))$:

- The chromatic number is exactly four.
- The non-identity component is connected.
- There are no cliques greater than three.
- There are cycles of every length.

Questions about 334-triangle graphs more generally:

- For $H \triangleleft G$: how are $\Delta 334(G)$, $\Delta 334(H)$ and $\Delta 334(G/H)$ related?

Read our Paper!

For more information and detailed proofs, see our paper:

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Eric S. Egge and Michaela A. Polley

Acknowledgments

For more information or any further questions, please contact me at `polleym@carleton.edu`.

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